

Importance of separated efficiencies between positively and negatively charged particles for cumulant calculations

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We show the importance of separated efficiency corrections between positively and negatively charged particles for cumulant calculations by Monte Carlo toy models and analytical calculations. Our results indicate that $S\sigma$ in published net-proton results from the STAR experiment will be suppressed about 5 to 10% in central collisions, and 10 to 20% in peripheral collisions at the beam energy of $\sqrt{s_{NN}} = 62.4$ and 200 GeV if the separated efficiencies are used to efficiency correction.

I. INTRODUCTION

A. Motivation

Cumulants of conserved quantities (net-charge, net-baryon or net-strangeness) are a powerful tool for searching the QCD critical point. Theoretically, cumulants are proportional to the power of correlation length and directly connected to the susceptibilities [1–3]. At the STAR experiment, cumulant ratios of net-proton and net-charge multiplicity distributions have been measured as a function of beam energy [4, 5]. The results of net-proton cumulants and their ratios suggest that there might be something interesting around $\sqrt{s_{NN}} = 20$ GeV, but more statistics is still necessary at low beam energy region due to their large statistical errors [4]. The results of net-charge are consistent with statistical baselines [5]. Charged particles are measured by the Time Projection Chamber (TPC). Tracking efficiency of the TPC for protons is about 70–90% at midrapidity ($|y| < 0.5$) in the range $0.4 < p_T < 0.8$ GeV/ c , which depends on centralities and beam energies. In order to take into account the effect of finite tracking efficiency on net-proton cumulants, efficiency corrections [6, 7] are utilized, which assume the binomial response of tracking efficiency. Efficiency correction formulas assuming the identical efficiency between positively and negatively charge particles shown in [6] was used in [4, 5]. However, there is about a few percent difference of tracking efficiency between positively and negatively charged particles. The main goal of this paper is to study how the published results will be changed if the separated efficiencies are used, which will be shown by Monte Carlo toy models and analytical calculations.

B. Cumulants and their baselines

At the STAR experiment, cumulants up to fourth order were measured, and cumulant ratios (which can be also written in terms of moments) defined as below were

measured as a function of beam energies,

$$S\sigma = \frac{C_3}{C_2} = \frac{\chi_3}{\chi_2}, \quad (1)$$

$$\kappa\sigma^2 = \frac{C_4}{C_2} = \frac{\chi_4}{\chi_2}, \quad (2)$$

where σ , S and κ are the second to fourth order moments respectively, and χ_n is the n -th order susceptibility [3]. As cumulants are extensive variables, the volume effects can be canceled by taking their ratios. Theoretically, these variables are predicted to diverge near around the QCD critical point [2]. Experimentally, these observables are compared with statistical baseline which is called Skellam distribution in order to find a non-monotonic signal [8]. Skellam distribution is the difference between two independent Poisson distributions. Odd and even order cumulants of Skellam distribution are expressed in terms of mean parameter of Poisson distribution.

$$C_{\text{odd}} = \mu_+ - \mu_-, \quad (3)$$

$$C_{\text{even}} = \mu_+ + \mu_-, \quad (4)$$

where μ_{\pm} is the mean parameter of Poisson distribution for positively and negatively charged particles. Then the statistical baselines of cumulant ratios can be expressed as

$$\left. \frac{C_3}{C_2} \right|_{\text{Skellam}} = \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-}, \quad (5)$$

$$\left. \frac{C_4}{C_2} \right|_{\text{Skellam}} = 1. \quad (6)$$

Note that the baseline of C_3/C_2 can change with centrality and beam energy, while the baseline of C_4/C_2 is unity by definition.

II. ANALYSIS

In this section, we discuss the potential problems of averaged efficiency correction by comparison with separated efficiency corrections. First, we show explicit expressions of cumulants for averaged and separated efficiency corrections in II A. Second, we demonstrate simple Monte Carlo toy models assuming Skellam distributions in II B. Finally, we explain the results from II B using analytical calculations in II C.

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A. Efficiency correction

In order to discuss the analytical formulas of efficiency correction, we use the recursive expressions of moments and cumulants,

$$\mu_n = \langle (M_+ - M_-)^n \rangle = \sum_{r=0}^n \binom{n}{r} (-1)^r \langle M_+^{n-r} M_-^r \rangle, \quad (7)$$

$$C_n = \mu_n - \sum_{r=1}^{n-1} \binom{n-1}{r-1} C_r \mu_{n-r}, \quad (8)$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \quad (9)$$

where μ_n is the n -th order non-central moment, C_n denotes the n -th order cumulant, M_{\pm} is the number of observed particles and brackets represent the average over

many events. Once non-central moments up to n -th order are calculated by Eq. (7), one can immediately calculate the n -th order cumulant recursively by Eq. (8).

Now let us define N_{\pm} as the number of produced particles, ε_{\pm} as the efficiency for positively and negatively charged particles. $K_{n,\text{ave}}$ and $K_{n,\text{sep}}$ are the n -th order cumulant corrected by using averaged efficiency or separated efficiencies, respectively. The first order cumulant can be simply corrected,

$$K_{1,\text{sep}} = \langle N_+ \rangle - \langle N_- \rangle = \frac{\langle M_+ \rangle}{\varepsilon_+} - \frac{\langle M_- \rangle}{\varepsilon_-}, \quad (10)$$

$$K_{1,\text{ave}} = \frac{\langle M_+ \rangle - \langle M_- \rangle}{\varepsilon} = \frac{C_1}{\varepsilon}, \quad (11)$$

$$\varepsilon = \frac{\varepsilon_+ + \varepsilon_-}{2}, \quad (12)$$

Equation (11) can be easily obtained from Eq. (10) by replacing ε_{\pm} to ε . Similarly, the second order cumulant can be written as

$$\begin{aligned} K_{2,\text{sep}} &= \mu_2 - \mu_1^2, \\ &= \langle N_+^2 \rangle - 2\langle N_+ N_- \rangle + \langle N_-^2 \rangle - \left(\langle N_+ \rangle - \langle N_- \rangle \right)^2, \\ &= \frac{\langle M_+^2 \rangle}{\varepsilon_+^2} - \frac{\langle M_+ \rangle}{\varepsilon_+^2} + \frac{\langle M_+ \rangle}{\varepsilon_+} - 2 \frac{\langle M_+ M_- \rangle}{\varepsilon_+ \varepsilon_-} + \frac{\langle M_-^2 \rangle}{\varepsilon_-^2} - \frac{\langle M_- \rangle}{\varepsilon_-^2} + \frac{\langle M_- \rangle}{\varepsilon_-} - \left(\frac{\langle M_+ \rangle}{\varepsilon_+} - \frac{\langle M_- \rangle}{\varepsilon_-} \right)^2. \end{aligned} \quad (13)$$

$$\begin{aligned} K_{2,\text{ave}} &= \frac{1}{\varepsilon^2} \left[\langle M_+^2 \rangle + \langle M_-^2 \rangle - 2\langle M_+ M_- \rangle - \left(\langle M_+ \rangle - \langle M_- \rangle \right)^2 \right] - \frac{1}{\varepsilon^2} \left(\langle M_+ \rangle + \langle M_- \rangle \right) (1 - \varepsilon), \\ &= \frac{1}{\varepsilon^2} \left[C_2 - n(1 - \varepsilon) \right], \end{aligned} \quad (14)$$

where $n = f_{10} + f_{01}$ and f_{ij} is the factorial moment (see VI A). From the second to third line, we used Eq. (46) shown in Appendix VI A. Equation (14) is the same as Eq. (17) shown in [6]. See Appendix VI C for the third order cumulant.

B. Monte Carlo toy model

Let us suppose two independent Poisson distributions, one is for positively charged particles and the other is for negatively charged particles, which are randomly generated according to parameters, $\mu_+ = 10$ and $\mu_- = 8$, as shown in Fig. 1 (a) and (b). These particles are randomly sampled by each binomial efficiency, $\varepsilon_+ = 0.66$ and $\varepsilon_- = 0.65$ as shown in Fig. 1 (c) and (d). Then, efficiency correction is applied by using averaged efficiency $\varepsilon = (\varepsilon_+ + \varepsilon_-)/2$ or separated efficiencies. Efficiency corrected cumulants become

$$K_{1,\text{ave}} = 2.14 \pm 0.02, \quad K_{1,\text{sep}} = 2.00 \pm 0.02 \quad (15)$$

where the simulation was performed by generating 100K events, and by using 30 independent trials to evaluate statistical uncertainties on cumulants. We can see that the first order cumulant is increased 7% from Skellam baseline ($K_1 = 2$) if we use averaged efficiency. Of course, the separated efficiency correction gives a consistent result with Skellam baseline. It is surprising that about 1%

difference of efficiencies leads to 7% deviation of C_1 from input value.

We focus on the relative deviation of corrected cumulants from the Skellam baselines, $(K_n - B_n)/B_n$, where K_n represents the n -th order corrected cumulant and B_n represents the Skellam baseline which is determined only by Poisson parameter as shown in Eqs. (3) and (4). This value should be zero if the efficiency correction works well. Figure 2 shows the relative deviation as a function of each order of cumulants and cumulant ratios, where the parameter of Poisson distribution μ_+ (μ_-) is the efficiency corrected first order cumulants of proton (antiproton) multiplicity distribution at (a) $\sqrt{s_{NN}} = 200$ GeV and (b) $\sqrt{s_{NN}} = 7.7$ GeV taken from [4]. Efficiency ε_{\pm} was taken from [9]. The simulation was performed by generating 100M events with 30 independent trials. The results of separated efficiency corrections are consistent with zero for all the order cumulants and cumulant ratios, which confirms the validity of this efficiency correction. In case of averaged efficiency at $\sqrt{s_{NN}} = 200$ GeV (see Fig. 2 (a)), however, K_1 and K_3 systematically increase about 10% from input value, while there is very small deviation for K_2 and K_4 . By contrast at $\sqrt{s_{NN}} = 7.7$ GeV (see Fig. 2 (b)), K_1 to K_4 increase about 2%. We will verify this observation by analytical calculations at next section.

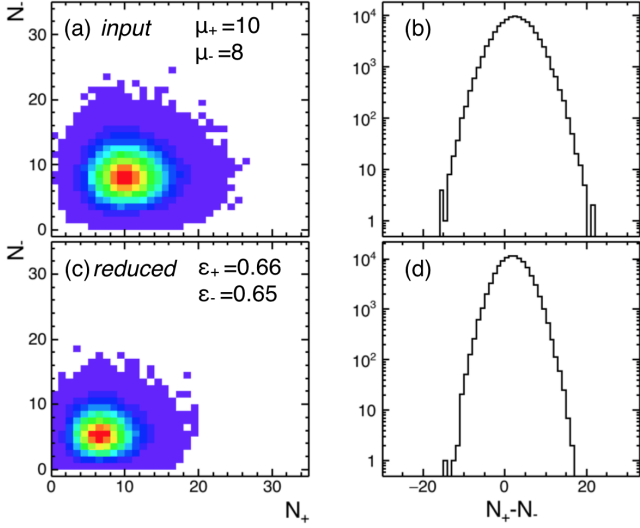


FIG. 1. (a) Correlation between positively and negatively charged particles. (b) Net-charge distribution which is calculated from (a). (c) Correlation between positively and negatively charged particles, which are randomly sampled from (a) according to binomial efficiency. (d) Net-charge distribution which is calculated from (c).

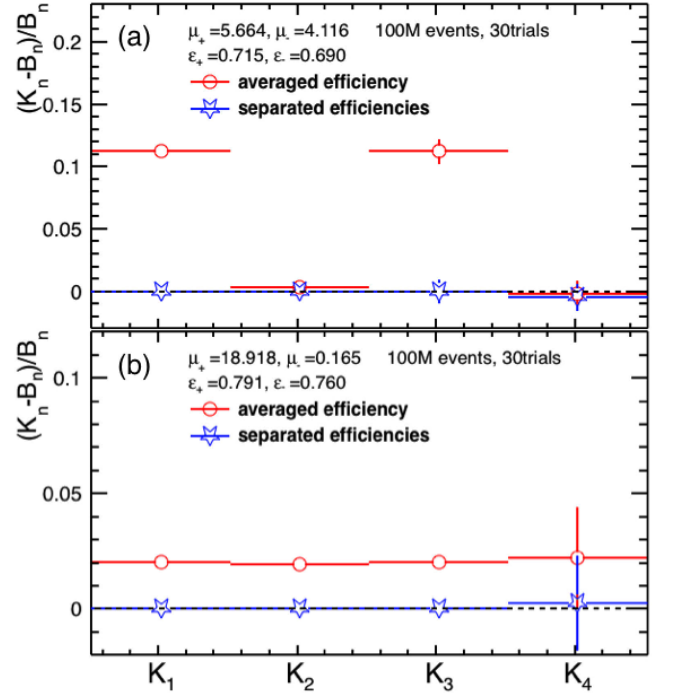


FIG. 2. (color online) Deviation of cumulants from input value assuming (a) $\sqrt{s_{NN}} = 200$ GeV and (b) $\sqrt{s_{NN}} = 7.7$ GeV, where μ_{\pm} and ε_{\pm} were taken from [4, 9]. K_n represents the n -th order corrected cumulant and B_n represents the Skellam baseline of the n -th order cumulant.

C. Analytical calculation

Now we show how the results of averaged efficiency deviates from input value for the first and second order cumulants. First, we rewrite Eq. (10) as below:

$$K_{1,\text{sep}} = \frac{\langle M_+ \rangle}{\varepsilon + \Delta\varepsilon} - \frac{\langle M_- \rangle}{\varepsilon - \Delta\varepsilon}, \quad (16)$$

where

$$\varepsilon = \frac{\varepsilon_+ + \varepsilon_-}{2}, \quad \Delta\varepsilon = \frac{\varepsilon_+ - \varepsilon_-}{2}.$$

We can apply Taylor expansion around $\Delta\varepsilon = 0$ to Eq. (10) under the assumption that $\Delta\varepsilon$ is negligible, we obtain

$$\begin{aligned} K_{1,\text{sep}}(\Delta\varepsilon) &\approx K_{1,\text{sep}}(0) + \left. \frac{\partial K_{1,\text{sep}}}{\partial \Delta\varepsilon} \right|_{\Delta\varepsilon=0} \Delta\varepsilon + \mathcal{O}(\Delta\varepsilon^2) \\ &\approx \frac{\langle M_+ \rangle - \langle M_- \rangle}{\varepsilon} - \frac{\langle M_+ \rangle + \langle M_- \rangle}{\varepsilon^2} \Delta\varepsilon \\ &= K_{1,\text{ave}} - \frac{\langle M_+ \rangle + \langle M_- \rangle}{\varepsilon^2} \Delta\varepsilon. \end{aligned} \quad (17)$$

Thus,

$$\begin{aligned} \Delta K_1 &= K_{1,\text{ave}} - K_{1,\text{sep}} \\ &\approx \frac{\Delta\varepsilon}{\varepsilon^2} (\langle M_+ \rangle + \langle M_- \rangle). \end{aligned} \quad (18)$$

Similarly for ΔK_2 ,

$$\begin{aligned}\Delta K_2 &= K_{2,\text{ave}} - K_{2,\text{sep}} \\ &\approx -\frac{2\Delta\varepsilon}{\varepsilon^2} \left[\frac{X_+ - X_-}{\varepsilon} - \frac{1}{2} (\langle M_+ \rangle - \langle M_- \rangle) \right],\end{aligned}\quad (19)$$

where $X_{\pm} = \langle M_{\pm} \rangle - \langle M_{\pm}^2 \rangle + \langle M_{\pm} \rangle^2$. See Appendix VIC for the analytical expression of ΔK_3 . From Eqs. (18) and (19), we can intuitively predict that the deviation of odd order cumulants is proportional to the sum of function of multiplicity, and the deviation of even order cumulants is proportional to the difference of function of multiplicity.

Since it is very cumbersome to calculate more than third order cumulant, we consider the general properties of difference between odd and even order cumulants. From the definitions of cumulants and non-central moments (see Eqs. (7) and (8)) and the analytical formula shown in Eqs. (10), (13) and (51), even order cumulants and non-central moments are *symmetrical* with M_+ and M_- , while odd orders are *antisymmetrical*. Therefore, they can be generally expressed as

$$C_{2m+1,\text{sep}} = f_{2m+1}(M_+, M_-) - f_{2m+1}(M_-, M_+), \quad (20)$$

$$C_{2m,\text{sep}} = f_{2m}(M_+, M_-) + f_{2m}(M_-, M_+), \quad (21)$$

where f_{2m+1} and f_{2m} are functions which satisfies above two equations for odd and even order cumulants respectively. Similarly we define the efficiency corrected cumulants as

$$K_{2m+1,\text{sep}} = F_{2m+1}[(M_+, \varepsilon_+), (M_-, \varepsilon_-)] - F_{2m+1}[(M_-, \varepsilon_-), (M_+, \varepsilon_+)], \quad (22)$$

$$K_{2m,\text{sep}} = F_{2m}[(M_+, \varepsilon_+), (M_-, \varepsilon_-)] + F_{2m}[(M_-, \varepsilon_-), (M_+, \varepsilon_+)], \quad (23)$$

where F is the efficiency corrected result of f , which can be expressed in terms of M_{\pm} and ε_{\pm} . Replacing ε_{\pm} to $\varepsilon \pm \Delta\varepsilon$, we obtain

$$\begin{aligned}K_{2m+1,\text{sep}} &\approx F_{2m+1}[(M_+, \varepsilon), (M_-, \varepsilon)] - F_{2m+1}[(M_-, \varepsilon), (M_+, \varepsilon)] \\ &+ \left[\frac{\partial F_{2m+1}[(M_+, \varepsilon + \Delta\varepsilon), (M_-, \varepsilon - \Delta\varepsilon)]}{\partial \Delta\varepsilon} \Big|_{\Delta\varepsilon=0} - \frac{\partial F_{2m+1}[(M_-, \varepsilon - \Delta\varepsilon), (M_+, \varepsilon + \Delta\varepsilon)]}{\partial \Delta\varepsilon} \Big|_{\Delta\varepsilon=0} \right] \Delta\varepsilon + \mathcal{O}(\Delta\varepsilon^2) \\ &\approx K_{2m+1,\text{ave}} \\ &+ \left[\frac{\partial F_{2m+1}[(M_+, \varepsilon + \Delta\varepsilon), (M_-, \varepsilon - \Delta\varepsilon)]}{\partial \Delta\varepsilon} \Big|_{\Delta\varepsilon=0} - \left(\frac{\partial \Delta\varepsilon'}{\partial \Delta\varepsilon} \right) \frac{\partial F_{2m+1}[(M_-, \varepsilon + \Delta\varepsilon'), (M_+, \varepsilon - \Delta\varepsilon')]}{\partial \Delta\varepsilon'} \Big|_{\Delta\varepsilon'=0} \right] \Delta\varepsilon,\end{aligned}\quad (24)$$

where

$$\Delta\varepsilon' = -\Delta\varepsilon. \quad (25)$$

Then, we define $G(x, y, \varepsilon, \Delta\varepsilon)$ as derivative of $F[(x, \varepsilon_+), (y, \varepsilon_-)]$ with respect to a ,

$$G(x, y, \varepsilon, \Delta\varepsilon) = \frac{\partial F[(x, \varepsilon + a), (y, \varepsilon - a)]}{\partial a} \Big|_{a=0} \Delta\varepsilon. \quad (26)$$

Therefore,

$$\begin{aligned}\Delta K_{2m+1} &= K_{2m+1,\text{ave}} - K_{2m+1,\text{sep}}, \\ &= G_{2m+1}(M_+, M_-, \varepsilon, \Delta\varepsilon) + G_{2m+1}(M_-, M_+, \varepsilon, \Delta\varepsilon).\end{aligned}\quad (27)$$

Similarly for even order cumulants,

$$\begin{aligned}\Delta K_{2m} &= K_{2m,\text{ave}} - K_{2m,\text{sep}}, \\ &= G_{2m}(M_+, M_-, \varepsilon, \Delta\varepsilon) - G_{2m}(M_-, M_+, \varepsilon, \Delta\varepsilon).\end{aligned}\quad (28)$$

Eqs. (27) and (28) indicate that the deviation of odd order cumulants is represented as sum of $G(M_+, M_-, \varepsilon, \Delta\varepsilon)$ and $G(M_-, M_+, \varepsilon, \Delta\varepsilon)$, while the deviation of even order cumulants is represented as difference between them. Confirmation of these calculations is shown in Appendix. VID. Note that Eqs. (27) and (28) are valid for any probability distribution.

Now let us assume that both positively and negatively charged particles follow Poisson distribution. From the definition of Skellam distribution shown in Eqs. (3) and

(4), we can easily derive the following expressions:

$$\Delta K_{\text{odd}} \approx \frac{\Delta\varepsilon}{\varepsilon^2} (\langle M_+ \rangle + \langle M_- \rangle), \quad (29)$$

$$\Delta K_{\text{even}} \approx \frac{\Delta\varepsilon}{\varepsilon^2} (\langle M_+ \rangle - \langle M_- \rangle). \quad (30)$$

TABLE I. Comparison of ΔK_n between toy model simulations and analytical calculations assuming Skellam distribution at $\sqrt{s_{NN}} = 200$ GeV.

n -th order	MC toy model	analytical calculation
1	0.1123 ± 0.0002	0.112
2	0.0028 ± 0.0002	0.004
3	0.112 ± 0.010	0.112
4	-0.002 ± 0.010	0.004

TABLE II. Comparison of ΔK_n between toy model simulations and analytical calculations assuming Skellam distribution at $\sqrt{s_{NN}} = 7.7$ GeV.

n -th order	MC toy model	analytical calculation
1	$0.0202 \pm 2.4 \times 10^{-5}$	0.021
2	0.0195 ± 0.0002	0.020
3	0.020 ± 0.002	0.021
4	0.022 ± 0.022	0.020

Equations (29) and (30) indicate that the deviation of odd order cumulant is proportional to the sum of multiplicity, while the deviation of even order cumulant is proportional to the difference of multiplicity. Using parameters shown in Fig. 2 (a) we can estimate ΔK_n analytically. Comparison between toy model simulations and analytical calculations are summarized in Tab. I and II. These results are roughly consistent, although ΔK_4 has large statistical errors.

III. RESULTS

As we discussed in previous section, the deviation of $K_{n,ave}$ depends on the multiplicity, which indicates that it also depends on the beam energies. Fig. 3 shows the deviation of (a) K_1 , K_2 and K_3 (b) $S\sigma$ and $\kappa\sigma^2$ (c) $S\sigma/Skellam$ from input value as a function of beam energy for central and peripheral collisions, where these deviations are defined as

$$\Delta K_{n,ave} = K_{n,ave} - B_n, \quad (31)$$

$$\Delta(S\sigma)_{ave} = (S\sigma)_{ave} - B_{S\sigma}, \quad (32)$$

$$\Delta(\kappa\sigma^2)_{ave} = (\kappa\sigma^2)_{ave} - B_{\kappa\sigma^2}, \quad (33)$$

$$\Delta(S\sigma/Skellam)_{ave} = (S\sigma/Skellam)_{ave} - B_{S\sigma/Skellam}, \quad (34)$$

where $K_{n,ave}$, $(S\sigma)_{ave}$, $(\kappa\sigma^2)_{ave}$ and $(S\sigma/Skellam)_{ave}$ denote the n -th order cumulant, $S\sigma$, $\kappa\sigma^2$ and $S\sigma/Skellam$ corrected by using averaged efficiency. B_n , $B_{S\sigma}$, $B_{\kappa\sigma^2}$ and $B_{S\sigma/Skellam}$ are their Skellam baselines. From Fig. 2 one can see that the separated efficiency correction gives a correct value which equals to the Skellam baseline. Thus, we used Skellam baselines in Eqs. (31)–(34) for simplicity. As one can see in Fig. 3 (a), the deviation decreases as beam energy, and different behavior can be observed between odd and even order as discussed in previous section. At high beam energies, the deviation of K_2 from input value is close to 0 while the deviations for K_1 and K_3 stays about 0.2 because net-baryon is very small at midrapidity, $\bar{p}/p \simeq 0.727$ at $\sqrt{s_{NN}} = 200$ GeV [4]. At low beam energies, the deviation of K_2 is as large as that

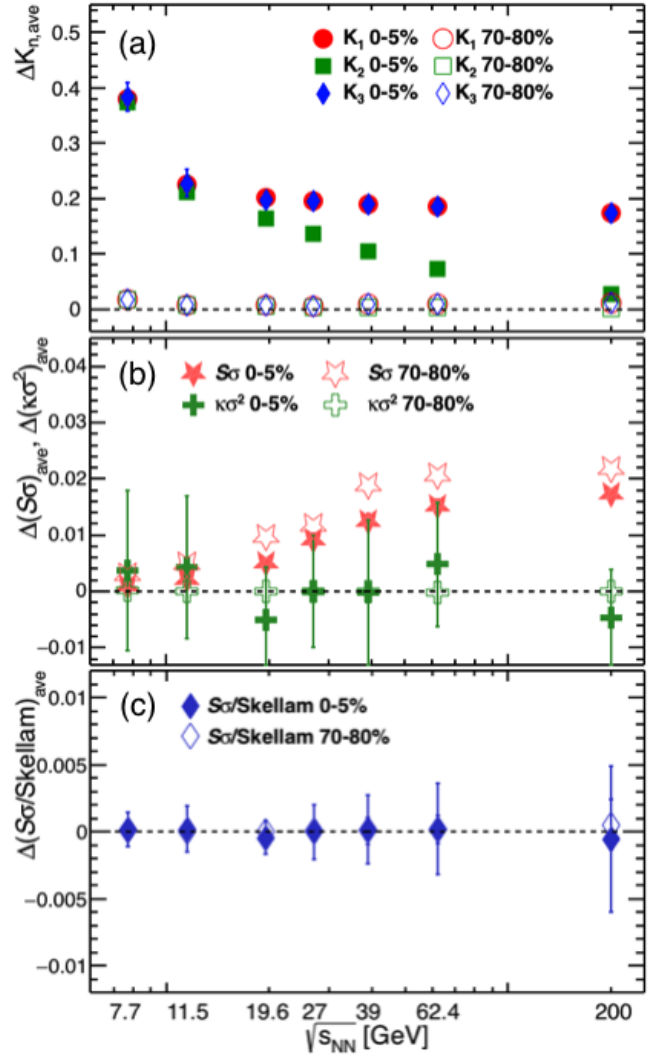


FIG. 3. (color online) Deviation of (a) K_1 , K_2 and K_3 (b) $S\sigma$ and $\kappa\sigma^2$ (c) $S\sigma/Skellam$ from input value as a function of beam energy. They are defined in Eqs. (31)–(34). Closed symbols represent the result assuming 0-5% central collisions, and open symbols represent the result assuming 70-80% peripheral collisions. Dashed lines represent the baseline for efficiency correction ($=0$).

of K_1 and K_3 due to large net-baryon, $\bar{p}/p \simeq 0.009$ at $\sqrt{s_{NN}} = 7.7$ GeV [4]. This difference between odd and even order leads to the behavior of $S\sigma$ and $\kappa\sigma^2$ shown in Fig. 3 (b). The deviation of $S\sigma$ increases as beam energy, and the deviation of $\kappa\sigma^2$ becomes almost zero, because $\kappa\sigma^2$ is the ratio of even order cumulants. As one can see in Fig. 3 (c), the deviation of $S\sigma/Skellam$ is zero for all over the beam energy. This is because the Skellam baseline of $S\sigma$ is also affected by averaged efficiency (see IV A for details).

In order to discuss the relative deviation as a function of beam energy, we also studied the beam energy dependence of Skellam baseline, which can be directly calculated according to Eqs. (3)–(6) without MC toy models. Figure 4 shows Skellam baselines of (a) odd and even order cumulant (b) $S\sigma$, $\kappa\sigma^2$ and $S\sigma/Skellam$ as a function of beam energy for central and peripheral collisions. We note that the baselines of $\kappa\sigma^2$ and $S\sigma/Skellam$ are unity

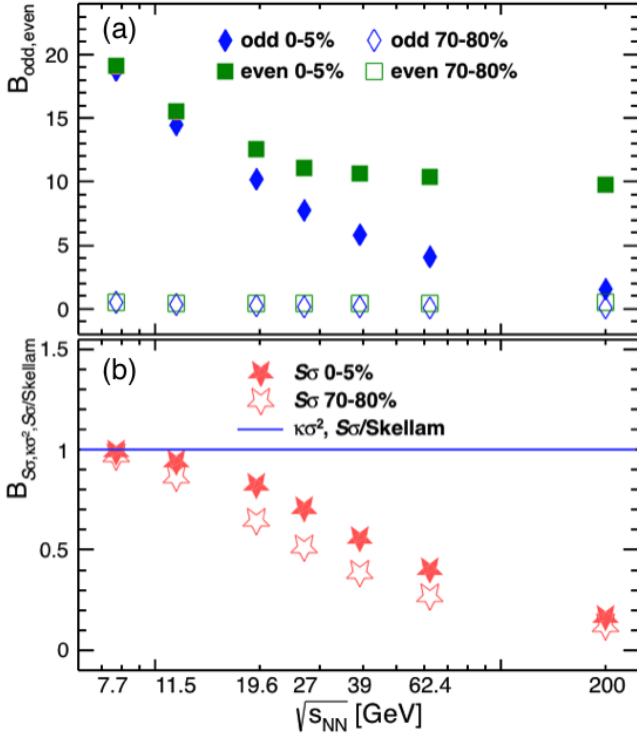


FIG. 4. (color online) Skellam baseline of (a) odd and even order cumulants (b) $S\sigma$, $\kappa\sigma^2$ and $S\sigma/Skellam$ as a function of beam energy. Closed symbols represent the result assuming 0-5% central collisions, and open symbols represent the result assuming 70-80% peripheral collisions. Baselines of $\kappa\sigma^2$ and $S\sigma/Skellam$ are represented as a solid blue line since they are unity by definition.

by their definitions.

Figure 5 shows the relative deviation of (a) K_1 , K_2 and K_3 (b) $S\sigma$ and $\kappa\sigma^2$ (c) $S\sigma/Skellam$ from input value as a function of beam energy, which corresponds to the ratio of Fig. 3 to Fig. 4. K_1 and K_3 deviate about 10 to 20% in peripheral collisions and 5 to 10% in central collisions at $\sqrt{s_{NN}} = 62.4$ and 200 GeV, while the deviation of K_2 is very small, which leads to the large deviation of $S\sigma$ and small deviation of $\kappa\sigma^2$ and $S\sigma/Skellam$. $S\sigma/Skellam$ doesn't deviate for all over the beam energy.

IV. DISCUSSION

A. $S\sigma/Skellam$

As can be seen in Fig. 2, $S\sigma$ deviates from input value because it is defined as the ratio of third to second order cumulant, nevertheless $S\sigma/Skellam$ doesn't deviate (see Figs. 3 (c) and 5 (c)). This is because Skellam term is also affected if we use the averaged efficiency. From Eq. (5),

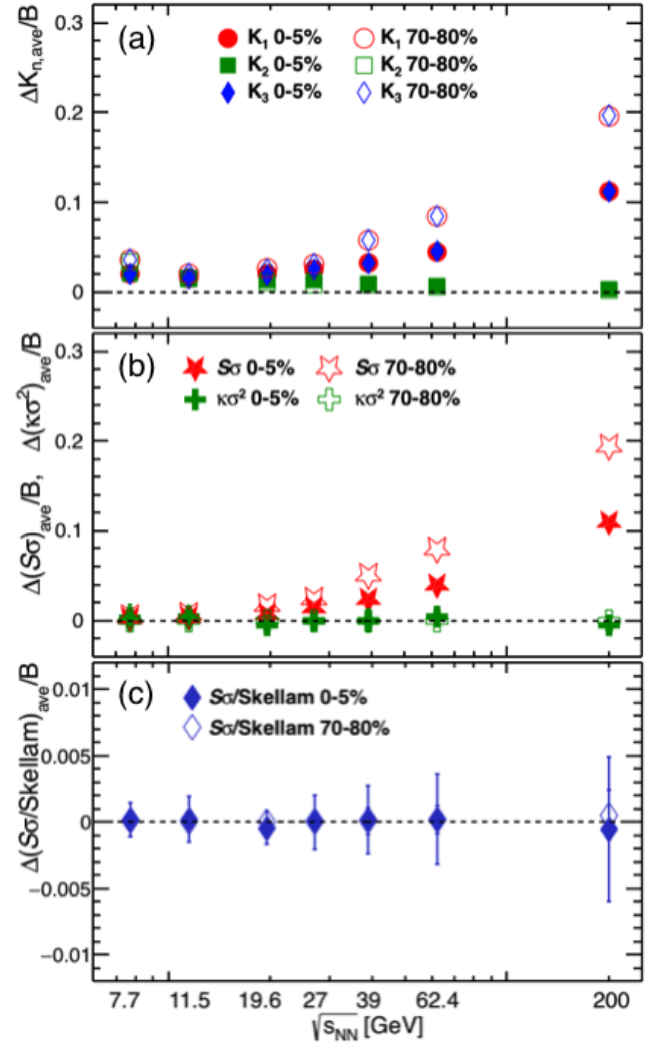


FIG. 5. (color online) Relative deviation of (a) K_1 , K_2 and K_3 (b) $S\sigma$ and $\kappa\sigma^2$ (c) $S\sigma/Skellam$ from input value as a function of beam energy, which corresponds to the ratio of Fig. 3 to Fig. 4. Closed symbols represent the result assuming 0-5% central collisions, and open symbols represent the result assuming 70-80% peripheral collisions. Dashed lines represent the baseline for efficiency correction ($=0$).

we obtain

$$\left. \frac{C_3}{C_2} \right|_{Skellam} = \frac{\langle M_+ \rangle - \langle M_- \rangle}{\langle M_+ \rangle + \langle M_- \rangle}, \quad (35)$$

$$\begin{aligned} \left. \frac{K_3}{K_2} \right|_{Skellam, \text{sep}} &= \frac{\langle N_+ \rangle - \langle N_- \rangle}{\langle N_+ \rangle + \langle N_- \rangle} = \frac{\frac{\langle M_+ \rangle}{\varepsilon_+} - \frac{\langle M_- \rangle}{\varepsilon_-}}{\frac{\langle M_+ \rangle}{\varepsilon_+} + \frac{\langle M_- \rangle}{\varepsilon_-}} \\ &= \frac{\varepsilon_- \langle M_+ \rangle - \varepsilon_+ \langle M_- \rangle}{\varepsilon_- \langle M_+ \rangle + \varepsilon_+ \langle M_- \rangle}. \end{aligned} \quad (36)$$

In case of averaged efficiency, we obtain

$$\begin{aligned} \left. \frac{K_3}{K_2} \right|_{Skellam,ave} &= \frac{\frac{\langle M_+ \rangle - \langle M_- \rangle}{\varepsilon}}{\frac{\langle M_+ \rangle + \langle M_- \rangle}{\varepsilon}}, \\ &= \frac{\langle M_+ \rangle - \langle M_- \rangle}{\langle M_+ \rangle + \langle M_- \rangle} = \frac{C_3}{C_2} \Big|_{Skellam}, \end{aligned} \quad (37)$$

which means the Skellam term is still uncorrected if we use the averaged efficiency. Thus, the reason why $S\sigma/Skellam$ doesn't deviate from input value is because numerator ($S\sigma$) and denominator (Skellam) are affected simultaneously.

B. Weighted averaged efficiency

Intuitively, it is more appropriate for averaged efficiency to be weighted by number of particles as

$$\varepsilon_w = \frac{\langle N_+ \rangle \varepsilon_+ + \langle N_- \rangle \varepsilon_-}{\langle N_+ \rangle + \langle N_- \rangle}, \quad (38)$$

where N_{\pm} is the number of particles. This *weighted averaged* efficiency should consider the effect of small number of anti-protons at low beam energy region. MC toy models using ε_w was also studied. Fig. 6 shows the relative deviation from input value in two cases, one is calculated by averaged efficiency, the other is calculated by weighted averaged efficiency as a function of beam energy assuming 0-5% central collisions. Weighted averaged efficiency gives better results than averaged efficiency for K_1 , K_2 and K_3 at low beam energy region due to small number of anti-protons, while the results of weighted averaged efficiency deviate as large as the averaged efficiency at high beam energy region. For $S\sigma$ the results of weighted averaged efficiency deviate as large as the averaged efficiency at all over the beam energy. These results indicate that we should not use the averaged efficiency as well as weighted one.

V. SUMMARY

Importance of separated efficiencies between positively and negatively charged particles was shown by Monte Carlo toy models and analytical calculations. MC toy models assuming parameters from [4] indicate that, odd order cumulants and $S\sigma$ systematically deviate from input value about 10% in case of averaged efficiency at $\sqrt{s_{NN}} = 200$ GeV, while the deviation of even order cumulants is as large as odd order cumulants at $\sqrt{s_{NN}} = 7.7$ GeV. In order to understand the nature of this behavior, analytical calculation was also performed. Results indicate that the deviation of odd order cumulants is proportional to the sum of multiplicity, while the deviation of even order cumulants is proportional to the difference of multiplicity. Moreover, beam energy dependence was studied assuming the published net-proton results. $S\sigma/Skellam$ and $\kappa\sigma^2$ don't deviate from input value for beam energies in which we studied. $S\sigma$ is enhanced about 5 to 10% in central collisions and 10 to 20%

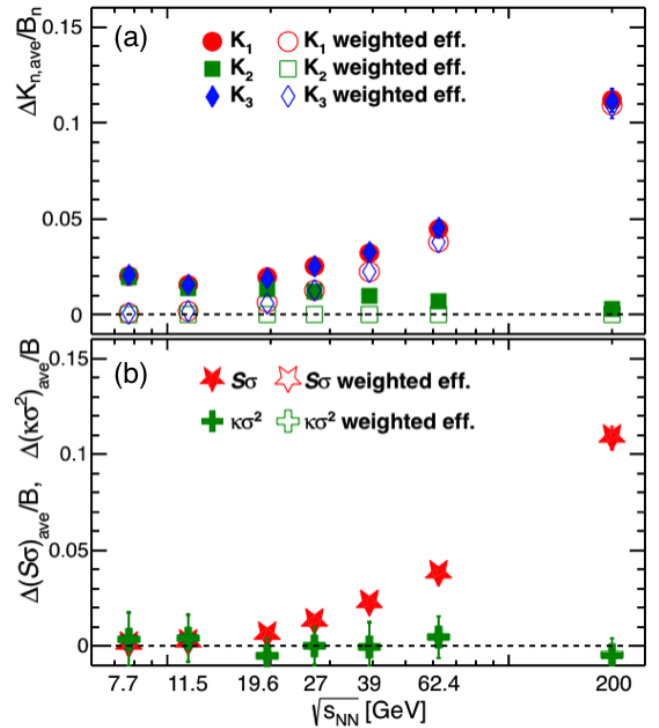


FIG. 6. (color online) Relative deviation of results corrected by using averaged efficiency and weighted averaged efficiency as a function of beam energy. Each panel shows (a) K_1 , K_2 and K_3 (b) $S\sigma$ and $\kappa\sigma^2$. For $S\sigma$ and $\kappa\sigma^2$ the results of weighted averaged efficiency are overlapped with the results of averaged efficiency. Dashed lines represent the baseline for efficiency correction ($=0$).

in peripheral collisions at $\sqrt{s_{NN}} = 62.4$ and 200 GeV, and there is less than 5% enhancement at $\sqrt{s_{NN}} = 7.7$ –39 GeV, where the QCD critical point is predicted to exist. Therefore, these enhancement arising from using the averaged efficiency don't change the conclusions in [4]. However, it is definitely the right way to use separated efficiencies, not the averaged one.

In order to develop correction methods which assumes more realistic situation of heavy ion colliding experiments, more detailed characteristics were studied for efficiency correction, i.e. correction formulas imposing phase space dependent efficiencies [10, 11], and the breaking of the binomial assumption of tracking efficiencies [12]. Recently at the STAR experiment, the p_T region are extended up to 2.0 GeV/c by including Time Of Flight for particle identification [9], which leads to the low tracking efficiency at high p_T region. This effect is corrected by dividing phase space of (anti)proton into high and low p_T regions using the efficiency correction from [13].

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VI. APPENDIXES

A. Factorial moments

Factorial moments have a useful characteristics as shown below,

$$f_{ab} = \varepsilon_+^a \varepsilon_-^b F_{ab}, \quad (39)$$

$$f_{ab} = \sum_{M_+=a}^{\infty} \sum_{M_-=b}^{\infty} p(M_+, M_-) \frac{M_+!}{(M_+ - a)!} \frac{M_-!}{(M_- - b)!}, \quad (40)$$

$$F_{ab} = \sum_{N_+=a}^{\infty} \sum_{N_-=b}^{\infty} P(N_+, N_-) \frac{N_+!}{(N_+ - a)!} \frac{N_-!}{(N_- - b)!}. \quad (41)$$

They can be also written in terms of Stirling number of the first kind.

$$f_{ab} = \sum_{i=0}^a \sum_{j=0}^b s(a, i) s(b, j) \langle M_+^i M_-^j \rangle, \quad (42)$$

$$F_{ab} = \sum_{i=0}^a \sum_{j=0}^b s(a, i) s(b, j) \langle N_+^i N_-^j \rangle, \quad (43)$$

thus,

$$\sum_{i=0}^a \sum_{j=0}^b s(a, i) s(b, j) \langle M_+^i M_-^j \rangle = \varepsilon_+^a \varepsilon_-^b \sum_{i=0}^a \sum_{j=0}^b s(a, i) s(b, j) \langle N_+^i N_-^j \rangle. \quad (44)$$

since $s(i, i) = 1$, one can deduce the following recursive expressions,

$$\langle M_+^a M_-^b \rangle + \sum_{\substack{i,j \geq 0 \\ i,j \neq a,b}} s(a, i) s(b, j) \langle M_+^i M_-^j \rangle = \varepsilon_+^a \varepsilon_-^b \langle N_+^a N_-^b \rangle + \sum_{\substack{i,j \geq 0 \\ i,j \neq a,b}} s(a, i) s(b, j) \langle N_+^i N_-^j \rangle, \quad (45)$$

$$\rightarrow \langle N_+^a N_-^b \rangle = \frac{\langle M_+^a M_-^b \rangle}{\varepsilon_+^a \varepsilon_-^b} + \sum_{\substack{i,j \geq 0 \\ i,j \neq a,b}} s(a, i) s(b, j) \left(\frac{\langle M_+^i M_-^j \rangle}{\varepsilon_+^i \varepsilon_-^j} - \langle N_+^i N_-^j \rangle \right), \quad (46)$$

where i and j are increased from 0 to a or b , but they cannot be a or b simultaneously. By using Eq.(46), one can express $\langle N_+^a N_-^b \rangle$ in terms of the combination of $\langle M_+^i M_-^j \rangle$ [14]. Moreover, one can apply efficiency corrections to the arbitrary order of cumulant using programming language.

B. Stirling number of the first kind

Stirling number of the first kind is defined as

$$s(n, k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}, \quad (47)$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = |s(n, k)|, \quad (48)$$

is the unsigned Stirling number of the first kind, which can be calculated by recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}, \quad (49)$$

with the initial conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0. \quad (50)$$

C. ΔK_3

Efficiency corrections of the third order cumulant are expressed as below:

$$\begin{aligned}
K_{3,\text{sep}} &= \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 \\
&= \langle N_+^3 \rangle - 3\langle N_+^2 N_- \rangle + 3\langle N_+ N_-^2 \rangle - \langle N_-^3 \rangle - 3(\langle N_+ \rangle - \langle N_- \rangle)(\langle N_+^2 \rangle - 2\langle N_+ N_- \rangle + \langle N_-^2 \rangle) + 2(\langle N_+ \rangle - \langle N_- \rangle)^3 \\
&= \frac{1}{\varepsilon_+^3} \left[\langle M_+^3 \rangle + 2\langle M_+ \rangle - 3\langle M_+^2 \rangle - 3\langle M_+ \rangle(\langle M_+^2 \rangle - \langle M_+ \rangle) + 2\langle M_+ \rangle^3 \right] \\
&\quad - \frac{1}{\varepsilon_-^3} \left[\langle M_-^3 \rangle + 2\langle M_- \rangle - 3\langle M_-^2 \rangle - 3\langle M_- \rangle(\langle M_-^2 \rangle - \langle M_- \rangle) + 2\langle M_- \rangle^3 \right] \\
&\quad - \frac{3}{\varepsilon_+^2 \varepsilon_-} \left[\langle M_+^2 M_- \rangle - \langle M_+ M_- \rangle - \langle M_- \rangle(\langle M_+^2 \rangle - \langle M_+ \rangle) - 2\langle M_+ \rangle \langle M_+ M_- \rangle + 2\langle M_+ \rangle^2 \langle M_- \rangle \right] \\
&\quad + \frac{3}{\varepsilon_-^2 \varepsilon_+} \left[\langle M_-^2 M_+ \rangle - \langle M_- M_+ \rangle - \langle M_+ \rangle(\langle M_-^2 \rangle - \langle M_- \rangle) - 2\langle M_- \rangle \langle M_- M_+ \rangle + 2\langle M_- \rangle^2 \langle M_+ \rangle \right] \\
&\quad + \frac{3}{\varepsilon_+^2} \left[\langle M_+^2 \rangle - \langle M_+ \rangle - \langle M_+ \rangle^2 \right] - \frac{3}{\varepsilon_-^2} \left[\langle M_-^2 \rangle - \langle M_- \rangle - \langle M_- \rangle^2 \right] + \frac{\langle M_+ \rangle}{\varepsilon_+} - \frac{\langle M_- \rangle}{\varepsilon_-}. \tag{51}
\end{aligned}$$

$$\begin{aligned}
K_{3,\text{ave}} &= \frac{1}{\varepsilon^3} \left[\langle M_+^3 \rangle + 2\langle M_+ \rangle - 3\langle M_+^2 \rangle - 3\langle M_+ \rangle(\langle M_+^2 \rangle - \langle M_+ \rangle) + 2\langle M_+ \rangle^3 \right] \\
&\quad - \frac{1}{\varepsilon^3} \left[\langle M_-^3 \rangle + 2\langle M_- \rangle - 3\langle M_-^2 \rangle - 3\langle M_- \rangle(\langle M_-^2 \rangle - \langle M_- \rangle) + 2\langle M_- \rangle^3 \right] \\
&\quad - \frac{3}{\varepsilon^3} \left[\langle M_+^2 M_- \rangle - \langle M_- \rangle \langle M_+^2 \rangle - 2\langle M_+ \rangle \langle M_+ M_- \rangle + 2\langle M_+ \rangle^2 \langle M_- \rangle \right] \\
&\quad + \frac{3}{\varepsilon^3} \left[\langle M_-^2 M_+ \rangle - \langle M_+ \rangle \langle M_-^2 \rangle - 2\langle M_- \rangle \langle M_- M_+ \rangle + 2\langle M_- \rangle^2 \langle M_+ \rangle \right] \\
&\quad + \frac{3}{\varepsilon^2} \left[\langle M_+^2 \rangle - \langle M_+ \rangle - \langle M_+ \rangle^2 \right] - \frac{3}{\varepsilon^2} \left[\langle M_-^2 \rangle - \langle M_- \rangle - \langle M_- \rangle^2 \right] + \frac{\langle M_+ \rangle}{\varepsilon} - \frac{\langle M_- \rangle}{\varepsilon}. \tag{52}
\end{aligned}$$

Similarly to ΔK_1 and ΔK_2 (see Eqs.(18) and (19)), we can obtain ΔK_3 as

$$\begin{aligned}
\Delta K_3 &= K_{3,\text{ave}} - K_{3,\text{sep}} \\
&\approx \left[\frac{1}{\varepsilon^4} \left[3(A_+ + A_-) - (B_+ + B_-) \right] + \frac{2}{\varepsilon^3} (C_+ + C_-) + \frac{2}{\varepsilon^2} (D_+ + D_-) \right] \Delta\varepsilon, \tag{53}
\end{aligned}$$

where constant terms are defined as

$$\begin{aligned}
A_{\pm} &= \langle M_{\pm}^3 \rangle + 2\langle M_{\pm} \rangle - 3\langle M_{\pm}^2 \rangle - 3\langle M_{\pm} \rangle(\langle M_{\pm}^2 \rangle - \langle M_{\pm} \rangle) + 2\langle M_{\pm} \rangle^3, \\
B_{\pm} &= 3\langle M_{\pm}^2 M_{\mp} \rangle - 3\langle M_{\pm} M_{\mp} \rangle - 3\langle M_{\mp} \rangle(\langle M_{\pm}^2 \rangle - \langle M_{\pm} \rangle) - 6\langle M_{\pm} \rangle \langle M_{\pm} M_{\mp} \rangle + 6\langle M_{\pm} \rangle^2 \langle M_{\mp} \rangle, \\
C_{\pm} &= 3(\langle M_{\pm}^2 \rangle - \langle M_{\pm} \rangle^2 - \langle M_{\pm} \rangle), \\
D_{\pm} &= \langle M_{\pm} \rangle.
\end{aligned}$$

D. Examples of general expression

In case of K_1 (see Eq. (18)), we obtain

$$\begin{aligned}
f(M_+, M_-) &= \langle M_+ \rangle, & f(M_-, M_+) &= \langle M_- \rangle, \\
F[(M_+, \varepsilon_+), (M_-, \varepsilon_+)] &= \frac{\langle M_+ \rangle}{\varepsilon_+}, & F[(M_-, \varepsilon_-), (M_+, \varepsilon_+)] &= \frac{\langle M_- \rangle}{\varepsilon_-}, \\
G(M_+, M_-, \varepsilon, \Delta\varepsilon) &= \frac{\Delta\varepsilon}{\varepsilon^2} \langle M_+ \rangle, & G(M_-, M_+, \varepsilon, \Delta\varepsilon) &= \frac{\Delta\varepsilon}{\varepsilon^2} \langle M_- \rangle. \tag{54}
\end{aligned}$$

Similarly for K_2 (see Eq. (19)),

$$\begin{aligned}
f(M_+, M_-) &= \langle M_+^2 \rangle - \langle M_+ M_- \rangle - \langle M_+^2 \rangle + \langle M_+ \rangle \langle M_- \rangle, \\
f(M_-, M_+) &= \langle M_-^2 \rangle - \langle M_- M_+ \rangle - \langle M_-^2 \rangle + \langle M_- \rangle \langle M_+ \rangle, \\
F[(M_+, \varepsilon_+), (M_-, \varepsilon_-)] &= \frac{\langle M_+^2 \rangle}{\varepsilon_+^2} - \frac{\langle M_+ \rangle}{\varepsilon_+^2} + \frac{\langle M_+ \rangle}{\varepsilon_+} - \frac{\langle M_+ M_- \rangle}{\varepsilon_+ \varepsilon_-} - \frac{\langle M_+ \rangle^2}{\varepsilon_+^2} + \frac{\langle M_+ \rangle \langle M_- \rangle}{\varepsilon_+ \varepsilon_-}, \\
F[(M_-, \varepsilon_-), (M_+, \varepsilon_+)] &= \frac{\langle M_-^2 \rangle}{\varepsilon_-^2} - \frac{\langle M_- \rangle}{\varepsilon_-^2} + \frac{\langle M_- \rangle}{\varepsilon_-} - \frac{\langle M_- M_+ \rangle}{\varepsilon_- \varepsilon_+} - \frac{\langle M_- \rangle^2}{\varepsilon_-^2} + \frac{\langle M_- \rangle \langle M_+ \rangle}{\varepsilon_- \varepsilon_+}, \\
G(M_+, M_-, \varepsilon, \Delta\varepsilon) &= -\frac{2\Delta\varepsilon}{\varepsilon^2} \left[\frac{X_+}{\varepsilon} - \frac{1}{2} \langle M_+ \rangle \right], \\
G(M_-, M_+, \varepsilon, \Delta\varepsilon) &= -\frac{2\Delta\varepsilon}{\varepsilon^2} \left[\frac{X_-}{\varepsilon} - \frac{1}{2} \langle M_- \rangle \right].
\end{aligned} \tag{55}$$

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